ION-ACOUSTIC TURBULENCE IN A COLLISIONLESS SHOCK WAVE

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UDC 533.951.7/.3

The structure of a collisionless shock wave at the front of which ion-acoustic turbulence is excited is investigated. On the basis of the theory of anomalous resistance, equations are obtained for the oscillational spectrum and the particle distribution function in the plasma which, when known, make it possible to determine the magnetic field profile, density, and other macroscopic characteristics of the shock wave. The possibility of comparing theoretical predictions with experimental results from light scattering at the shock front is discussed.

1. Introduction. It can be considered an established fact that the existence of collisionless shock waves in a plasma within a magnetic field is related to the phenomenon of anomalous resistance over a broad range of parameters [1]. The reason for its formation is strong plasma nonuniformity and consequent excitation of various instabilities. A large number of experimental results related to shock waves propagated across a magnetic field show that in a low-pressure plasma such that $\beta \equiv 8\pi p/H^2 \ll 1$ (p is the gaskinetic plasma pressure, H is magnetic field intensity) and in relatively weak magnetic fields (where the electron plasma frequency ω_{pe} is much greater than the electron cyclotron frequency ω_{He}), ion-acoustic instability plays a leading role. Its development is well described by the theory of weak turbulence, and the basic problem encountered here is the choice of mechanism providing establishment of an equilibrium level of fluctuation. This question has been discussed many times, and an analysis of experimental data [2] makes it possible to give preference to linear Landau damping by ions [3].

Knowing the energy of the oscillations, one can determine from quasilinear equations the effective collision frequency of the particles, i.e., the dissipative plasma properties which determine shock-wave structure. Since the time for instability development, which is equal to the inverse of the growth rate, γ^{-1}, in order of magnitude, is much less than the time for the passage of the front through a given point in space ($\tau \sim \delta/u$, where δ is the width of the front, and u is the velocity of shock-wave propagation), one first determines local particle distribution functions established by the effects of quasilinear collisions as is done in gas dynamics. They are far from Maxwellian as will be seen subsequently. On the basis of conservation laws one can then obtain magnetohydrodynamic equations describing the variation of macroscopic plasma characteristics in space and time.

2. Electron and Ion Distribution Functions and Vibrational Spectra. Ordinarily, the electron-scattering frequency in shock waves is considerably less than the cyclotron frequency. Therefore, the directed electron motion is a drift motion. Let it occur along the x axis at some velocity $\bar{v}_e$ (we assume the magnetic field is directed along z). Then the electron velocity distribution function $f_e(\mathbf{v})$ in the rest system of the electrons is axially symmetric around the z axis. In the present case where electron scattering is associated with ion-acoustic oscillations, it can be assumed generally isotropic: $F_e(v)$. This is a consequence of the fact that the electrons are scattered nearly elastically because of the small phase velocity of ion sound. Therefore, the x axis – the direction of electron drift – is a uniquely preferred direction for the excitation of vibrations.

We introduce spherical coordinates in velocity space (v, θ, φ) and in wave vector space (k, θ', φ') with polar axis along x. We denote the electrostatic energy density of the oscillations by $W(k, \theta')$. According to the definition of the quasilinear diffusion coefficients for electrons, we have

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 38-43, May-June, 1974. Original article submitted November 9, 1972.

$$D_{\alpha\beta}^{(e)} = \frac{8\pi^2 e^2}{m_e^2}\int \frac{k_\alpha k_\beta}{k^2} W_{\mathbf{k}} \delta(\omega_{\mathbf{k}} - \mathbf{kv})\, d^3\mathbf{k} \tag{2.1}$$

In this case, the only nonzero components are $D_{vv}^{(e)}$, $D_{v\theta}^{(e)}$, $D_{\theta\theta}^{(e)}$, and $D_{\varphi\varphi}^{(e)}$. In their calculation, one must bear in mind that in the electron rest system, which is moving with a velocity $\bar{V}_e$ that is much greater than the phase velocity of ion sound, the frequency of all vibrations (because of the Doppler effect) is $\omega k \approx -k_x\bar{V}_e = -k\bar{V}_e \cos\theta'$. In addition, we allow for the fact $\bar{V}_e$ is small in comparison with the thermal velocities of the electrons. We then obtain from Eq. (2.1)

$$D_{\theta\theta}^{(e)} = \frac{D(\theta)}{v \sin^2\theta}, \quad D_{v\theta}^{(e)} = \frac{D(\theta)}{v \sin\theta}\frac{\bar{V}_e}{v}, \quad D_{vv}^{(e)} = \frac{D(\theta)}{v}\frac{\bar{V}_e^2}{v^2}$$

$$D(\theta) = \frac{16\pi^2 e^2}{m_e^2}\int_{\pi/2-\theta}^{\pi/2} \frac{\sin\theta' \cos^2\theta'\, d\theta'}{[\sin^2\theta - \cos^2\theta']^{1/2}}\int_0^\infty W(k,\theta')\, dk \tag{2.2}$$

for $\theta < \pi/2$, $D(\pi-\theta) = D(\theta)$.

An equation for the isotropic electron distribution function $F_e(v)$ is found by integration of the quasilinear equation over the angle θ,

$$\frac{dF_e}{\partial t} = \frac{1}{2}\int_0^\pi \sin\theta\, d\theta \left[\frac{\partial}{\partial v_\alpha} D_{\alpha\beta}^{(e)} \frac{\partial F_e}{\partial v_\beta}\right] = \frac{\bar{V}_e^2}{2v^2}\frac{\partial}{\partial v}\frac{1}{v}\frac{\partial F_e}{\partial v}\int_0^\pi D(\theta)\sin\theta\, d\theta \tag{2.3}$$

The electron growth rate for the excitation of vibrations determined by the function $F_e(v)$ is approximately

$$\gamma_e = \pi^2 \frac{m_i}{m_e}\frac{\omega_k^3}{k^2}\frac{F_e(0)}{n}\bar{V}_e \cos\theta' \tag{2.4}$$

We calculate the frictional force R experienced by the electrons during drift motion.

$$R = \int m\mathbf{v}\left[\frac{\partial}{\partial v_\alpha} D_{\alpha\beta}^{(e)}\frac{\partial F_e}{\partial v_\beta}\right] d^3\mathbf{v} = -2\pi m_e \bar{V}_e F_e(0)\int_0^\pi D(\theta)\sin\theta\, d\theta \tag{2.5}$$

Defining the effective electron scattering frequency ν_e so that $R = -m_e\bar{V}_e n \nu_e$, we obtain

$$\nu_e = 2\pi \frac{F_e(0)}{n}\int_0^\pi D(\theta)\sin\theta\, d\theta \tag{2.6}$$

We turn to a determination of the ion-velocity distribution. In the model assumed, the main mass of ions does not interact with the vibrations [3]. A contribution to linear Landau damping is made only by a small group of ions which may be in resonance with the vibrations. We introduce their distribution function $f_i(v, \theta)$ and denote their total number per unit volume by nx_i so that

$$\int f_i d^3\mathbf{v} = nx_i$$

The kinetic equation for f_i has the form

$$\frac{\partial f_i}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v} v^2 \left(D_{vv}^{(i)}\frac{\partial f_i}{\partial v} + \frac{D_{v\theta}^{(i)}}{v}\frac{\partial f_i}{\partial \theta}\right) + \frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\left(D_{v\theta}^{(i)}\frac{\partial f_i}{\partial v} + \frac{D_{\theta\theta}^{(i)}}{v}\frac{\partial f_i}{\partial\theta}\right) \tag{2.7}$$

(the specific form of the diffusion coefficients $D_{\alpha\beta}^{(i)}$ will be given below in more convenient variables).

Under the influence of the vibrations, the particle distribution functions vary so that the total growth rate $\gamma = \gamma_e + \gamma_i$ is close to zero for those values of the wave vector **k**, where $W_k \neq 0$ and is not positive where $W_k = 0$.

Writing an expression for the ion decrement

$$\gamma_i = \pi\frac{\omega_k^3}{nk^2}\int \frac{\omega}{k}\frac{\partial f_i}{\partial v} + \left(\frac{\omega}{kv}\cos\theta - \cos\theta'\right)\frac{\partial f_i}{\partial\theta}\left[\sin^2\theta\sin^2\theta' - \left(\frac{\omega}{kv} - \cos\theta\cos\theta'\right)^2\right]^{-1/2} d\theta dv \tag{2.8}$$

and considering that the electrons excite all waves for which $\cos\theta' \geq 0$, we obtain still another equation which the functions $f_i(v, \theta)$ must satisfy:

$$\gamma_i + \gamma_e = \begin{cases} 0 & \text{for} \quad \cos\theta' \geqslant 0 \\ \leqslant 0 & \text{for} \quad \cos\theta' < 0 \end{cases} \tag{2.9}$$

The unknown functions $F_e(v, t)$, $f_i(v, \theta, t)$, and $W(k, \theta', t)$ must also be determined from Eqs. (2.1), (2.3), (2.7), and (2.9). These equations possess the property that F_e, f_i, and W cease to depend on initial conditions, and their evolution in time takes on a universal nature (in the terminology of [4], an asymptotic solution is established). Since the plasma is cold ($\beta_0 \ll 1$) ahead of the shock wave in the initial state, its heating occurs very rapidly, and a transition to the asymptotic solution occurs everywhere within the shock front. To determine the solution it is necessary to introduce in the equation self-similar variables, as has been done previously [4]. As a result, the electron distribution function

$$F_e = \frac{an}{(T_e/m_e)^{3/2}} \exp\left[-\frac{v^5}{b\,(T_e/m_e)^{5/2}}\right]$$
$$a = \frac{3}{4\pi 5^{3/5}[\Gamma(3/5)]^{5/2}}, \quad b = \left[5\Gamma\left(\frac{8}{5}\right)\right]^{5/2} \tag{2.10}$$

is determined where the temperature T_e is defined so that the electron kinetic energy density is $3nT_e/2$, and the pressure, correspondingly, is nT_e. The spectral density of the vibrational electrostatic energy is conveniently written in the form

$$W(k, \theta') = \frac{T_e^{3/2} m_i^{1/2}}{32\pi^2 e^2 \omega_{pe}^2 m_e^{3/2}} \frac{dT_e}{dt} w(q, \theta')$$
$$q = kT_e^{1/2}/m_e^{1/2}\omega_{pe} \tag{2.11}$$

The ion distribution function is

$$f_i = \frac{nx_i}{(T_i/m_i)^{3/2}} g_i(\xi, \theta)$$
$$\xi = m_i^{1/2} v / T_i^{1/2}, \quad 2\pi \int_0^{\pi} \sin\theta d\theta \int_0^{\infty} g_i(\xi, 0)\xi^2 d\xi = 1 \tag{2.12}$$

and the quantity T_i is the effective temperature of the resonance ions. An equation for $g_i(\xi, \theta)$ follows from Eq. (2.7):

$$\frac{\partial}{\partial \xi}\left(\xi^3 g_i + D_{\xi\xi}\frac{\partial g_i}{\partial \xi} + D_{\xi\theta}\frac{\partial g_i}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\left(D_{\xi\theta}\frac{\partial g_i}{\partial \xi} + D_{\theta\theta}\frac{\partial g_i}{\partial\theta}\right) = 0$$
$$D_{\xi\xi} = \frac{1}{\xi}\int \frac{\omega_q^2}{q}\Phi dq d\theta', \quad D_{\xi\theta} = \frac{1}{\xi}\int \omega_q\left(\frac{\omega_q}{q\xi}\operatorname{ctg}\theta - \frac{\cos\theta'}{\sin\theta}\right)\Phi\, dq d\theta'$$
$$D_{\theta\theta} = \frac{1}{\xi}\int q\left(\frac{\omega_q}{q\xi}\operatorname{ctg}\theta - \frac{\cos\theta'}{\sin\theta}\right)^2 \Phi dq d\theta' \tag{2.13}$$
$$\Phi = \frac{w(q, \theta')\sin\theta'}{[\sin^2\theta\sin^2\theta' - (\omega_q/q\xi - \cos\theta\cos\theta')^2]^{1/2}}$$
$$\omega_q = \omega_k/\omega_{pi}$$

In the asymptotic mode, the number of resonance ions x_i, the ratio of electron and ion temperature T_i/T_e, and the ratio of the drift velocity $\bar{V}_e$ to electron thermal velocity $(T_e/m_e)^{1/2}$ remain constant.

It was found [3] that

$$\bar{V}_e = \alpha_1 (m_e/m_i)^{1/4}(T_e/m_e)^{1/2}, \quad x_i = \alpha_2 (m_e/m_i)^{1/4}, \quad T_i = \alpha_3 T_e \tag{2.14}$$

where the numbers $\alpha_{1,2,3}$ are constants of the order of unity, and it is necessary to know the exact solution of Eq. (2.13) in order to determine them.

3. Structure of a Weak Shock Wave. In the present problem, dispersion effects associated with the inclusion of electron inertia or breakdown of quasineutrality are unimportant. It is therefore sufficient to limit ourselves to a single-fluid approximation. The plasma, as a whole, moves along the y axis – the direction of shock-wave propagation. We denote the plasma velocity by u. We calculate all quantities in a frame of reference moving with the shock wave. The electromagnetic field has the following nonzero components: $H_z = H(y)$; $E_y(y)$ is the electric field which arises because of plasma polarization, and E_x is the induced electric field in a wave system independent of the coordinates.

In the unperturbed plasma ahead of the shock wave, i.e., for $y \to +\infty$, the magnetic field is H_0, the density n_0, and the velocity u_0. The initial temperature can be assumed to be zero. The shock-wave intensity is characterized by the Mach number $M = u_0/v_A$, where $v_A = H_0/\sqrt{4\pi m_i n_0}$. We consider the case of weak shock waves for which $(M-1) \ll 1$. From the Hugoniot relations it follows that for this limit the change in the quantities H, n, and u in the shock wave are of the order of $(M-1)$, whereas the final electron temperature behind the shock front $T_e \sim (M-1)^3$. All the equations written below are simplified by the inclusion of these conditions.

Since there is no electric field in the laboratory system in the unperturbed plasma,

$$E_x = u_0 H_0 / c = M H_0 v_A / c \tag{3.1}$$

The polarization field E_y is set up so that the electrons and ions move together along y. Hence,

$$E_y = \overline{V}_e H(y) / c \tag{3.2}$$

The total force acting on the electrons in the drift direction must be equated to zero:

$$R = ne(E_x - uc^{-1}H) = nec^{-1}(Mv_A H_0 - uH) \tag{3.3}$$

The constancy of momentum flux density, the continuity equation, and the Maxwell equation for the magnetic field are written in the form

$$m_i n_0 M v_A (u - M v_A) + \frac{H^2}{8\pi} = \frac{H_0^2}{8\pi}, \quad nu = -n_0 M v_A, \quad \frac{dH}{dy} = -\frac{4\pi}{c} ne\overline{V}_e \tag{3.4}$$

Only Joule dissipation is important in the plasma thermal balance:

$${}^3/_2 nu dT_e/dy = R\overline{V}_e = ne\overline{V}_e c^{-1}(Mv_A H_0 - uH) \tag{3.5}$$

Considering that in weak shock waves the electron current velocity is proportional to their thermal velocity (2.14), we obtain from Eqs. (3.4) and (3.5) a single equation for the magnetic field profile which is written with the required accuracy in the form

$$\frac{c^2}{\omega_{pe}^2}\left(\frac{m_i}{m_e}\right)^{1/2}\frac{d^2h}{dy^2} = \frac{\alpha_1^2}{6}[4(M-1)h - 3h^2], \quad h = (H - H_0)/H_0$$

It is integrated to

$$\frac{h}{2(M-1)} = 1 - \mathrm{th}^2\left\{\frac{\alpha_1}{2}\left[\frac{2}{3}(M-1)\right]^{1/2}\frac{\omega_{pe}}{c}\left(\frac{m_e}{m_i}\right)^{1/4} y\right\} \tag{3.6}$$

Only that range of the coordinate y is of interest in which h varies from zero to $h_1 = {}^4/_3(M-1)$ – the value of the magnetic field in the final state behind the shock wave. As is clear from Eq. (3.6), the width δ of the weak shock front is the following:

$$\delta \sim \frac{c}{\omega_{pe}\sqrt{M-1}}\left(\frac{m_i}{m_e}\right)^{1/4} \tag{3.7}$$

The theoretically predicted constant ratio between electron current and thermal velocities is observed experimentally and the coefficient $\alpha \approx 1$–2 [2] from measurements in a hydrogen plasma. A comparison of the magnetic field profile (3.6) with measurement indicates that the approximation to a weak shock wave is applicable up to Mach numbers $M \le 2$. The particle distribution functions in the shock front are determined from Eqs. (2.10) and (2.12), and the oscillational spectra from Eq. (2.11), where $dT_e/dt = -u dT_e/dy$.

Further increase in shock-wave intensity leads to electron heating such that the quantity β in the shock front becomes of the order of unity. Electron thermal conductivity becomes important along with resistance. As a result, the electron velocity distribution function is "distorted" and the self-similar solution (2.10)-(2.14) becomes inapplicable. Formally, the numbers vary within the shock front, remaining quantities of the order of unity. Therefore, the order of magnitude of all quantities characterizing a shock wave is unchanged (in particular, their dependence on ion mass).

4. Oscillational Spectra. Experiments on light scattering at small angles in a shock front [5] are of great interest for an explanation of the mechanism for absorption of ion-acoustic vibrations. The scattering intensity is proportional to the Fourier components of the fluctuations in electron density produced during ion-acoustic turbulence. They are related to the spectral vibrational function W_k by the expression

$$\langle \delta n_k^2 \rangle \approx 10 \frac{n^2 e^2}{T_e^{3/2}} \frac{W_k}{k^2} \tag{4.1}$$

The quantity $\langle \delta n_k^2 \rangle$ calculated from the angular distribution of the scattered light [5] is nearly proportional to $1/k^3$ for $k\overline{r}_0 \le 1$, and falls rapidly with further increase in k (here $\overline{r}_0$ is some average value of the Debye radius in the shock front). The authors interpret this as the result of nonlinear Landau damping of ion sound by ions [6]. Since the diameter of the laser beam in these experiments was greater than the width of the front, only information about quantities averaged over the entire front is obtained from the scattering. Since the electron temperature and the Debye radius vary strongly within the front, the relationship given is of a qualitative nature.

In the present case, where damping by resonance ions occurs, one has not managed to determine W_k, and thereby $\langle \delta n_k^2 \rangle$, exactly. We shall try to estimate the dependence on $|k|$ roughly. To do this, we neglect angular dependence completely. The condition $\gamma = 0$ leads to $g_i(\xi) \sim 1/\xi$. We then obtain from Eq. (2.13)

$$D_{\xi\xi} \sim \frac{1}{\xi} \int w(q) \frac{\omega_q^2}{q} dq \sim \xi^4, \quad \omega_q = \frac{q}{(1+q^2)^{1/2}}$$

Here the integration is carried out over the range of values q where the phase velocity ω_q/q is less than ξ. This equation is solved with respect to w(q):

$$w(q) \sim (1 + q^2)^{-5/2}$$

Transforming to the usual variables, we have

$$W_k \sim (1 + k^2 r_0^2)^{-5/2}, \ \langle \delta n_k^2 \rangle \sim k^{-2} (1 + k^2 r_0^2)^{-5/2} \tag{4.2}$$

This result is in qualitative agreement with an earlier one [5].

The author thanks R. Z. Sagdeev and D. D. Ryutov for discussions of the work.

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